Another note on Dilworth's theorem in the infinite case

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In 1963, Perles [3] observed that, for an infinite cardinal κ, the product $\kappa \times \kappa$ is a poset without infinite antichains that cannot be covered by fewer than κ many chains. This shows that Dilworth's Theorem [2] fails very badly for posets with infinite width. In 1987, Abraham [1] showed that these are essentially the only posets with this property. Specifically, let's generalize Perles's counterexample as follows.

Definition. *A poset P is of* κ-Perles type *if there is a triangular array $\langle p_{\zeta,\xi} : \zeta \leq \xi < \kappa \rangle$ of elements of P such that if $\zeta < \zeta'$ and $\xi > \xi'$ then $p_{\zeta,\xi}$ and $p_{\zeta',\xi'}$ are incomparable.*

Clearly, $\kappa \times \kappa$ is a poset of κ-Perles type. It is not difficult to show (as I will below) that a poset of κ-Perles type cannot be covered by fewer than κ many chains. Abraham's Theorem can then be stated precisely.

Abraham's Theorem. *If P is a poset without infinite antichains that cannot be covered by κ many chains, then P is of κ^+-Perles type.*

In this note, I will present a proof of Abraham's Theorem. In [1], Abraham only proves the case $\kappa = \omega$, and he uses a slightly different characterization of Perles's counterexample. However, the proof of the more general result uses no fundamentally new ideas.

Abraham's original proof is very clever, but it is difficult to understand since it involves a complex mixed induction on the antichain rank and the cofinality of the poset. The proof presented below simplifies this by completely separating the antichain rank induction and the cofinality induction.

In the process, I will also isolate an interesting intermediate result — the Main Lemma.

For the remainder of this note, P is a poset, $\prec$ is the partial order on P, and $\perp$ is the incomparability relation on P. (The symbol $<$ is reserved for the order of ordinals.) For $\Box$ among $\prec, \succ, \preceq, \succeq, \perp$ and their negations, $P[\Box\, p]$ denotes the set $\{q \in P : q \Box p\}$. Similarly, if $Q \subseteq P$, we write $Q[\Box\, p]$ for $Q \cap P[\Box\, p]$.

First, let's see how posets of κ-Perles type do generalize Perles's original counterexample.

Perles's Theorem. *If P is of κ-Perles type then P cannot be covered by fewer than κ many chains. In particular, $\kappa \times \kappa$ cannot be covered by fewer than κ many chains.*

Proof. The result is clear when $\kappa = \omega$ since we can easily find antichains of arbitrarily large finite size. So suppose that $P = \bigcup_{\alpha<\mu} C_\alpha$ where $\omega \leq \mu < \kappa$. We need to show that some C_α contains two incomparable points.

Let $\langle p_{\zeta,\xi} : \zeta \leq \xi < \mu^+ \rangle$ be a triangular array witnessing that P is of μ^+-Perles type (which it is since $\mu^+ \leq \kappa$). For each $\zeta < \mu^+$, there is an $\alpha(\zeta) < \mu$ such that $\{\xi < \mu^+ : p_{\zeta,\xi} \in C_{\alpha(\zeta)}\}$ is unbounded in μ^+. Then there are $\zeta < \zeta' < \mu^+$ such that $\alpha(\zeta) = \alpha(\zeta') = \alpha$. We can then pick $\xi' < \xi < \mu^+$ such that $p_{\zeta,\xi}, p_{\zeta',\xi'} \in C_\alpha$. Now, $\zeta < \zeta'$ and $\xi > \xi'$, which implies that $p_{\zeta,\xi} \perp p_{\zeta',\xi'}$. Therefore, C_α is not a chain. □

Note that the proof of Perles's Theorem does not depend on the ordering of P at all. In fact, it applies more generally to any graph G: if there is a triangular array $\langle v_{\zeta,\xi} : \zeta \leq \xi < \kappa \rangle$ of vertices such that $\zeta < \zeta'$ and $\xi' < \xi$ imply that $v_{\zeta,\xi}$ and $v_{\zeta',\xi'}$ are independent, then G cannot be covered by fewer than κ many cliques.

The key to the proof of Abraham's Theorem is the following result.

Main Lemma. *If P is a poset such that $P[\not\succ p]$ can be covered by κ many chains for every $p \in P$, then either P can be covered κ many chains, or else P is of κ^+-Perles type.*

The proof of the Main Lemma is by induction on the cofinality of P. Although all of the principal ideas of the proof of the Main Lemma can be found in Abraham's proof, Abraham uses $P[\preceq p]$ instead of $P[\not\succ p]$ in the hypothesis and, to compensate for this weaker hypothesis, he also requires that P has no infinite antichains. This is ultimately how the Main Lemma will be used in the proof of Abraham's Theorem, but the more general formulation shows potential for more applications.

The proof of Abraham's Theorem from the Main Lemma uses induction on the antichain rank of P. This is discussed at length by in [1], so I will only briefly describe it here. If P has no infinite antichains, then the set A_P of all antichains of P is well-founded when ordered by reverse inclusion. (It is also well-founded when ordered by inclusion, but this is not as interesting.) Let ark_P be the rank function on A_P, i.e.,

$$\mathrm{ark}_P(a) = \sup\{\mathrm{ark}_P(b) + 1 : a \subsetneq b\}.$$

The antichain rank of $\mathrm{ark}(P)$ is then defined by $\mathrm{ark}(P) = \mathrm{ark}_P(\varnothing)$. An important fact to note is that

$$\mathrm{ark}(P[\perp p]) = \mathrm{ark}_P(\{p\}) < \mathrm{ark}_P(\varnothing) = \mathrm{ark}(P)$$

for every $p \in P$.

1 Proof of the Main Lemma

The proof is by induction on the cofinality of P, which is the smallest size of a set $Q \subseteq P$ with the property that $P = \bigcup_{q \in Q} P[\preceq q]$. The result is trivial for posets of cofinality 1, so suppose P is a poset of cofinality $\lambda > 1$, and that the result is known for all smaller cofinalities. There are then two cases depending on the cofinality of λ, as an ordinal. Note that $\mathrm{cof}(\lambda) \leq \kappa$ exactly when λ is the union of κ sets of size strictly less than λ.

Case $\mathrm{cof}(\lambda) \leq \kappa$. We can then write $P = \bigcup_{\alpha<\kappa} P_\alpha$ where each P_α has cofinality less than λ. By the induction hypothesis, each P_α is covered by κ many chains, and gathering these chains together we obtain a cover of P with κ many chains.

Case $\mathrm{cof}(\lambda) > \kappa$. Fix an enumeration $\langle r_\xi : \xi < \lambda \rangle$ of a cofinal subset of P. Without loss of generality, we may assume that if $\xi' < \xi$, then $r_\xi \not\preceq r_{\xi'}$. For $\xi < \lambda$, define:

$$a(p) = \min\{\xi < \lambda : p \preceq r_\xi\},$$

$$b(p) = \min\{\xi < \lambda : p \not\succ r_\xi\}.$$

Note that $a(r_\xi) = \xi$ for every $\xi < \lambda$. Let's record the following basic facts, which are easy to prove.

Lemma 1.1. *$b(p) \leq a(p)$.*

Lemma 1.2. *If $a(p) < b(q)$ then $p \prec q$.*

Lemma 1.3. *If $p \preceq q$ then $a(p) \leq a(q)$ and $b(p) \leq b(q)$.*

Lemma 1.4. *If $a(p) > a(q)$ and $b(p) < b(q)$ then $p \perp q$.*

Facts 1.1 and 1.4 suggest how the values of $a(p)$ and $b(p)$ may be related to the ξ and ζ coordinates of a triangular array as required for posets of Perles type. This is indeed the way that $a(p), b(p)$ will be used in Lemma 1.5. Fact 1.2 will be useful in Lemma 1.6.

For $\beta < \lambda$, define

$$R(\beta) = \{a(p) : b(p) = \beta\}.$$

Note that $R(\beta) \subseteq [\beta, \lambda)$ by Fact 1.1. Then consider the partition $\lambda = B \cup U$ where:

$$B = \{\beta < \lambda : \sup R(\beta) < \lambda\},$$
$$U = \{\beta < \lambda : \sup R(\beta) = \lambda\}.$$

We will now show that P can be covered by κ many chains when $|U| \leq \kappa$, and that P is of κ^+-Perles type when $|U| > \kappa$. The Main Lemma follows immediately.

Lemma 1.5. *If $|U| > \kappa$, then P is of κ^+-Perles type.*

Proof. Let $\langle \beta_\zeta : \zeta < \kappa^+ \rangle$ enumerate the first κ^+ elements of U. We will define a triangular array $\langle p_{\zeta,\xi} : \zeta \leq \xi < \kappa^+ \rangle$ of elements of P such that $b(p_{\zeta,\xi}) = \beta_\zeta$, and $\xi < \xi'$, then $a(p_{\zeta,\xi}) < a(p_{\zeta',\xi'})$ for all $\zeta < \xi$ and $\zeta' < \xi'$. This array witnesses that P is of κ^+-Perles type. Indeed, if $\zeta' < \zeta$ and $\xi < \xi'$, then $a(p_{\zeta,\xi}) < a(p_{\zeta',\xi'})$ and $b(p_{\zeta,\xi}) > b(p_{\zeta',\xi'})$. Therefore, $p_{\zeta,\xi} \perp p_{\zeta',\xi'}$ by Fact 1.4.

To construct the array, we proceed by induction on $\xi < \kappa^+$. Suppose we have defined $p_{\zeta',\xi'}$ for all $\zeta' \leq \xi' < \xi$. We will now define $p_{\zeta,\xi}$ by induction on $\zeta \leq \xi$.

First, let $\alpha_0 = \sup\{a(p_{\zeta',\xi'}) : \zeta' \leq \xi' < \xi\}$ and pick $p_{0,\xi}$ such that $b(p_{0,\xi}) = \beta_0$ and $a(p_{0,\xi}) > \alpha_0$. We can always do this since $R(\beta_0)$ is unbounded in λ and λ has cofinality greater than κ.

Next, suppose that $1 \leq \zeta \leq \xi$ and that we have defined $p_{\zeta',\xi}$ for $\zeta' < \zeta$. Let $\alpha_\zeta = \sup\{a(p_{\zeta',\xi}) : \zeta' < \zeta\}$ and pick $p_{\zeta,\xi}$ so that $b(p_{\zeta,\xi}) = \beta_\zeta$ and $a(p_{\zeta,\xi}) > \alpha_\zeta$. Again, we can always do this since $R(\beta_\zeta)$ is unbounded in λ and λ has cofinailty greater than κ.

This completes the construction, the properties are easily verified. □

Lemma 1.6. *If $|U| \leq \kappa$, then P can be covered by κ many chains.*

Proof. Note that since

$$\{p : b(p) = \beta\} \subseteq P[\not\succ p_\beta],$$

we see that $P_U = \{p : b(p) \in U\}$ can be covered by κ many chains. So it suffices to show that the complement of P_U, namely the set $P_B = \{p : b(p) \in B\}$, can also be covered by κ many chains.

Let Γ be the closed unbounded set of all $\gamma < \lambda$ such that $\sup R(\beta) < \gamma$ for every $\beta \in B \cap \gamma$, and let $\langle \gamma_\zeta : \zeta < \theta \rangle$ be the natural enumeration of Γ. (Note that $\gamma_0 = 0$.) For each $\zeta < \theta$, let

$$Q_\zeta = \{p : \gamma_\zeta \leq b(p) \leq a(p) < \gamma_{\zeta+1}\}.$$

Clearly, $P_B \subseteq \bigcup_{\zeta<\theta} Q_\zeta$. Also note that Q_ζ has cofinality at most $\gamma_{\zeta+1} < \lambda$. So, by the induction hypothesis, $Q_\zeta = \bigcup_{\alpha<\kappa}^{\infty} C_{\zeta,\alpha}$ where each $C_{\zeta,\alpha}$ is a chain.

Note that if $\zeta < \xi$ and $p \in Q_\zeta$, $q \in Q_\xi$, then $p \prec q$ by Fact 1.2, since

$$a(p) < \gamma_{\zeta+1} \leq \gamma_\xi \leq b(q).$$

It follows that each $C_\alpha = \bigcup_{\zeta<\theta} C_{\zeta,\alpha}$ is a chain, and $P_B \subseteq \bigcup_{\alpha<\kappa} C_\alpha$. □

2 Proof of Abraham's Theorem

The proof is by induction on the antichain rank of P, as described above. Suppose we know that Abraham's Theorem is true for posets of antichain rank less than α and that P is a poset of antichain rank α, which is not of κ^+-Perles type.

As previously observed, for every $p \in P$, we have $\text{ark}(P[\perp p]) < \text{ark}(P)$. Moreover, $P[\perp p]$ can't be of κ^+-Perles type. Thus, by the inductive hypothesis, we obtain the following fact.

Lemma 2.1. *For every $p \in P$, $P[\perp p]$ can be covered by κ many chains.*

If we had a way to cover P with at most κ many sets of the form $P[\perp p]$, then the result would follow immediately. For example, if P has a maximal chain of size at most κ, then we would be done. Unfortunately, there is no obvious reason why this would be the case. So we need to work a bit more.

Let's first narrow the types of posets we need to consider. The incomparability relation $\perp$ defines a graph on P, namely, the incomparability graph of P. Note that if Q_1, Q_2 are distinct connected components of this graph, then either $Q_1 \prec Q_2$ or $Q_1 \succ Q_2$. Thus P can be covered by κ many chains if and only if each connected component of the incomparability graph can be covered by κ many chains. We may thus assume that the incomparability graph of P is connected.

The Main Lemma tells us that we only need to show that $P[\not\succ p]$ can be covered by κ many chains for each $p \in P$. Observe that

$$P[\not\succ p] = \{p\} \cup P[\perp p] \cup P[\prec p].$$

So, by Lemma 1, we really only need to show that $P[\prec p]$ can be covered by κ many chains. It turns out that this observation is a little misleading. The correct way to view this decomposition is that $\{p\}$ is the set of elements of distance 0 from p in the incomparability graph of $P[\not\succ p]$, $P[\perp p]$ are the elements of distance 1 from p in the incomparability graph of $P[\not\succ p]$, and finally $P[\prec p]$ are the elements of distance at least 2 from p in the incomparability graph of $P[\not\succ p]$.

So let's fix $p_0 \in P$, and write D_n for the set of elements of distance n from p_0 in the incomparability graph of $P[\not\succ p_0]$. We only assumed that the incomparability graph of P was connected, but it is easy to see that the incomparability graph of $P[\not\succ p_0]$ is also connected; so this is indeed a decomposition of $P[\not\succ p_0]$.

Now we want to show that each D_n can be covered by κ many chains. This is clear for $D_0 = \{p_0\}$ and $D_1 = P[\perp p_0]$. For D_2, this is not immediately clear. By the dual of the Main Lemma, it suffices to show that $D_2[\not\prec q]$ can be covered by κ many chains for each $q \in D_2$. This is not immediately obvious, but Abraham has a neat trick to do this.

Again, it suffices to show that $D_2[\succeq q]$ can be covered by κ many chains. By definition of D_2, there is a $p \in D_1$ such that $p \perp q$. Clearly, $D_2[\succeq q] \cap P[\preceq p] = \varnothing$. Less clearly, $D_2[\succeq q] \cap P[\succeq p] = \varnothing$. (If $r \succeq p$, then $r \perp p_0$ and so $r \notin D_2$.) Therefore, $D_2[\succeq q] \subseteq P[\perp p]$ and hence $D_2[\succeq q]$ can be covered by κ many chains by Lemma 2.1. Therefore, D_2 can also be covered by κ many chains by the dual of the Main Lemma.

The heart of the previous trick is the fact that if $p \in D_1$ and $q \in D_2$ are incomparable, then $D_2[\not\prec q] \subseteq P[\perp p] \cup P[\perp q]$. This is also true for $p \in D_0$ and $q \in D_1$. In fact, it is true in general.

Lemma 2.2. *If $q \in D_{n+1}$ and $p \in D_n$ are incomparable, then*

$$D_{n+1}[\not\preceq q] \subseteq P[\perp p] \cup P[\perp q].$$

Proof. We show that $D_{n+1}[\succeq q] \subseteq P[\perp p]$. Again it is clear that $D_{n+1}[\succeq q] \cap P[\preceq p] = \varnothing$. Let's show that $D_{n+1}[\succeq q] \cap P[\succeq p] = \varnothing$. By definition of D_n, there is a sequence

$$p_0 \perp p_1 \perp \cdots \perp p_{n-1} \perp p_n = p.$$

Suppose that $p \preceq r$. Since $p_{n-1} \perp p_n = p$, we cannot have $r \preceq p_{n-1}$, so it must be the case that either $p_{n-1} \perp r$ or $p_{n-1} \preceq r$. In the case $p_{n-1} \perp r$, the distance from p_0 to r is at most n, so $r \notin D_{n+1}$. In the case $p_{n-1} \preceq r$, we can repeat the same argument to conclude that either $p_{n-2} \perp r$ or $p_{n-2} \preceq r$. Repeating as necessary, we see that either $r \in D_i$ for some $i \leq n$, or else $p_0 \preceq r$. In any case, we conclude that $r \notin D_{n+1}$.

We have shown that $D_{n+1}[\succeq q] \cap P[\succeq p]$ and $D_{n+1}[\succeq q] \cap P[\preceq p]$ are both empty. So it must be the case that $D_{n+1}[\succeq q] \subseteq P[\perp p]$. □

By Lemma 2.1 and 2.2, we see that $D_{n+1}[\not\preceq p]$ can be covered by κ many chains for every $p \in D_{n+1}$. It follows from the dual of the Main Lemma that each D_n can be covered by κ many chains. Hence, $P[\not\succeq p_0]$ can be covered by $\kappa \cdot \omega = \kappa$ many chains. Since p_0 was arbitrary, it follows from the Main Lemma that P can also be covered by κ many chains. This concludes the proof of Abraham's Theorem.

References

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